

There exists no distance-regular graph with intersection array $\{56, 36, 9; 1, 3, 48\}$

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Abstract

We prove that a distance-regular graph with intersection array $\{56, 36, 9; 1, 3, 48\}$ does not exist. This intersection array is from the table of feasible parameters for distance-regular graphs in "Distance-regular graphs" by A.E. Brouwer, A.M. Cohen, A. Neumaier.

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1 Introduction

The purpose of this paper is to prove the next

Theorem 1.1 *The array $\{56, 36, 9; 1, 3, 48\}$ cannot be realized as the intersection array of a distance-regular graph.*

This array is from the table of feasible parameters for distance-regular graphs in "Distance-regular graphs" by A.E. Brouwer, A.M. Cohen, A. Neumaier (see [1, p. 429]).

The important tool of our theorem's proof is the Koolen-Park inequality (see next section). This inequality shows the largest coclique in the vertex neighborhood of hypothetical distance-regular graph Γ with intersection array from Theorem 1.1 has size 3 (i.e., Γ does not contain a 4-claw). Using this observation, the possible neighborhoods of vertices of Γ are determined. For each of them, we will get a contradiction.

2 Definitions and preliminaries

We consider only finite undirected graphs without loops or multiple edges. Let Γ be a connected graph. The *distance* $d(u, w)$ between any two vertices u and w of Γ is the length of a shortest path from u to w in Γ . The *diameter* $\text{diam}(\Gamma)$ of Γ is the maximal distance occurring in Γ .

For a subset A of the vertex set of Γ , we will also write A for the subgraph of Γ induced by A . For a vertex u of Γ , define $\Gamma_i(u)$ to be the set of vertices that are at distance i from u ($0 \leq i \leq \text{diam}(\Gamma)$). The subgraph $\Gamma_1(u)$ is called the *neighborhood* of a vertex u (and it will be simply denoted as $\Gamma(u)$) and the *degree* of u is the number of neighbors of u , i.e., $|\Gamma(u)|$. A graph is *regular* with degree k if the degree of each of its vertices is k .

For the vertices $u_1, u_2, \dots, u_s \in \Gamma$, define $\Gamma(u_1, u_2, \dots, u_s)$ be the set of vertices of $\cap_{i=1}^s \Gamma(u_i)$. For two vertices $u, w \in \Gamma$ with $d(u, w) = 2$, the subgraph $\Gamma(u, w)$ is called the *μ -subgraph* of vertices u, w .

A connected graph Γ with diameter $d = \text{diam}(\Gamma)$ is *distance-regular* if there are integers b_i, c_i ($0 \leq i \leq d$) such that, for any two vertices $u, w \in \Gamma$ with $d(u, w) = i$, there are exactly c_i neighbors of w in $\Gamma_{i-1}(u)$ and b_i neighbors of w in $\Gamma_{i+1}(u)$ (we assume that $\Gamma_{-1}(u)$ and $\Gamma_{d+1}(u)$ are empty sets). In particular, a distance-regular graph Γ is regular with degree $b_0, c_1 =$

1 and $c_2 = \mu(\Gamma)$. For each vertex $u \in \Gamma$ and $0 \leq i \leq d$, the subgraph $\Gamma_i(u)$ is regular with degree $a_i = b_0 - b_i - c_i$. The numbers a_i, b_i, c_i ($0 \leq i \leq d$) are called the *intersection numbers* and the array $\{b_0, b_1, \dots, b_{d-1}; c_1, c_2, \dots, c_d\}$, is called the *intersection array* of the distance-regular graph Γ .

A *c-clique* C of Γ is a complete subgraph (i.e., every two vertices of C are adjacent) of Γ with exactly c vertices. We say that C is a clique if it is a c -clique for certain c . A coclique C of Γ is an induced subgraph of Γ with empty edge set. We say a coclique is a *c-coclique* if it has exactly c vertices.

The following lemma is due to J.H. Koolen and J. Park [2] (see also [3]).

Lemma 2.1 *Let Γ be a distance-regular graph and, for a vertex $x \in \Gamma$, the neighborhood of x contains a coclique of size $c \geq 2$. Then*

$$c_2 - 1 \geq \frac{c(a_1 + 1) - b_0}{\binom{c}{2}}.$$

3 A proof of theorem

Let Γ be a distance-regular graph with intersection array $\{56, 36, 9; 1, 3, 48\}$. The intersection number a_1 of Γ equals $56 - 36 - 1 = 19$. Fix an arbitrary vertex ∞ of Γ and denote the subgraph $\Gamma(\infty)$ by Δ . In particular, the graph Δ is regular of degree 19 and, for each pair of nonadjacent vertices x, y of Δ , $|\Delta(x, y)| \leq 2$ holds.

Lemma 3.1 *The largest coclique of Δ has size 3. Moreover, each vertex of Δ belongs to a maximal coclique of size 3.*

Proof. Let Δ contain a c -coclique. It is easy to see that each vertex of Δ belongs to a coclique of size at least $b_0/(a_1 + 1) = 56/20 > 2$. Hence, we may assume $c \geq 3$. If $c = 4$, then, by Lemma 2.1, $3 - 1 \geq (4(19 + 1) - 56)/6 = 4$, a contradiction. ■

Let the vertices $x_1, x_2, x_3 \in \Delta$ induce a 3-coclique. Denote the vertex set of $\Delta(x_i) - \cup_{j \neq i} \Delta(x_j)$ by X_i .

Lemma 3.2 *Without loss of generality, one of the following cases holds.*

- (1) $\Delta(x_1, x_2) = \{u, w\}$, $\Delta(x_1, x_3) = \{p\}$ and $\Delta(x_2, x_3) = \{q\}$,
- (2) $\Delta(x_1, x_2) = \{u, w\}$ and $\Delta(x_2, x_3) = \{p, q\}$,
- (3) $\Delta(x_1, x_2, x_3) = \{u\}$, $\Delta(x_1, x_2) = \{u, p\}$, $\Delta(x_2, x_3) = \{u, q\}$,
- (4) $\Delta(x_1, x_2, x_3) = \{u, w\}$.

Proof. For each pair of distinct indices $i, j \in \{1, 2, 3\}$, we denote $|\Delta(x_i, x_j) - \Delta(x_1, x_2, x_3)|$ by δ_{ij} and $|\Delta(x_1, x_2, x_3)|$ by δ . Then we have $|X_i| = 19 - \sum_{j, j \neq i} \delta_{ij} - \delta$ and $|\Delta| = 3 + \delta + \sum_{i < j} \delta_{ij} + \sum_i |X_i|$. Hence, $60 - 2\delta - \sum_{i < j} \delta_{ij} = 56$ and $2\delta + \sum_{i < j} \delta_{ij} = 4$. If $\delta = 0$, then either $\delta_{ij} = 2$ for two pair of indices (and we have Case (2)) or $\delta_{ij} = 1$ or 2 and we have Case (1). If $\delta = 1$, then $\delta_{ij} = 1$ for two pair of indices and we have Case (3). If $\delta = 2$, then $\delta_{ij} = 0$ and we have Case (4). The lemma is proved. ■

Lemma 3.3 *The following hold.*

- (1) X_i is a clique;
- (2) For a vertex z of $\Delta(x_i, x_j)$, either $X_i \subset \Delta(z)$ or $|X_i \cap \Delta(z)| \leq 1$ holds.

Proof. (1) If X_i contains a 2-coclique $\{a, b\}$, then the vertex set of $\cup_{j \neq i} \{x_j\} \cup \{a, b\}$ induce a 4-coclique in Δ . This contradicts Lemma 3.1.

(2) Suppose that $1 < |X_i \cap \Delta(z)| < |X_i|$. Then, for a vertex $y \in X_i - \Delta(z)$, the μ -subgraph of z, y contains the vertices ∞, x_i and $|X_i \cap \Delta(z)| \geq 2$ vertices of X_i , which is impossible. ■

Lemma 3.4 *Let $\{a, b\}$ be an edge of Δ such that $|\Delta(a, b)| \leq 1$, c be a vertex of $\Gamma_2(\infty) \cap \Gamma(a, b)$ and d be a vertex of $\Gamma(\infty, c) - \{a, b\}$. Then $d \in \Delta(a) \cup \Delta(b)$ and if $d \not\sim b$, then $\Delta(b, d) = \{a\}$.*

Proof. The subgraph $\Delta - \Delta(a) \cup \Delta(b)$ contains at most $56 - 2 \cdot 17 - 2 - 1 = 19$ vertices. Since $d(\infty, c) = 2$, $\Gamma(\infty, c)$ contains a, b and one more vertex, say, d . If $d \not\sim a$ and $d \not\sim b$, then $|\Gamma_3(c) \cap \Gamma(\infty)| \leq 56 - (2 \cdot 17 + 1 + (19 - 2)) < b_2 = 9$, a contradiction. If $d \in \Delta(a) - \Delta(b)$, then $\Gamma(b, d) = \{a, c, \infty\}$. ■

Lemma 3.5 *Case (4) is impossible.*

Proof. Note that $|X_i| = 17$ for $i = 1, 2, 3$. By Lemma 3.3(2), we may suppose $X_1 \subset \Delta(u)$. Since $|\Gamma(u, w)| \geq 4$, the vertices u and w are adjacent. Then u is adjacent to 17 vertices of X_1 and to 4 vertices w, x_1, x_2, x_3 , hence, $|\Delta(u)| > 19$, a contradiction. ■

Lemma 3.6 *Case (1) is impossible.*

Proof. We note that $|X_1| = |X_2| = 16$ and $|X_3| = 17$.

Let us first consider the case, when $X_1 \subset \Delta(p)$, $X_2 \subset \Delta(q)$. We may assume that $X_1 \subset \Delta(u)$. Then $p \sim u$, $X_2 \subset \Delta(w)$ and $q \sim w$.

Let y_1 be a vertex of X_1 . Then y_1 is adjacent to 18 vertices of $X_1 \cup \{x_1, p, u\}$, hence, y_1 is adjacent to a vertex of $X_2 \cup X_3$. Therefore, there are exactly 16 edges between X_1 and $X_2 \cup X_3$.

Let y_3 be a vertex of X_3 . Then y_3 is adjacent to 17 vertices of $X_3 \cup \{x_3\}$, hence, y_3 is adjacent to a couple of vertices of $X_1 \cup X_2$. Since $|\Delta(p, y_3)| \leq 2$, the vertex y_3 has exactly one neighbor in X_1 and exactly one neighbor in X_2 . This implies that there are 17 edges between X_1 and X_3 , which is impossible.

We may now suppose $X_3 \subset \Delta(p)$. Then $X_2 \subset \Delta(q)$ and $p \not\sim q$.

Suppose that $X_1 \subset \Delta(u)$, $X_2 \subset \Delta(w)$. Then $q \sim w$, $w \not\sim u$ and u is adjacent to a vertex of X_3 . Let y_1 be a vertex of X_1 . Then y_1 is adjacent to 17 vertices of $X_1 \cup \{x_1, u\}$, hence, y_1 is adjacent to a couple of vertices of $X_2 \cup X_3$. Because $\Delta(y_1, x_2)$ contains u and $\Delta(y_1, p)$ contains x_1 , the vertex y_1 has exactly one neighbor in X_2 and exactly one neighbor in X_3 . Let y_2 be a vertex of X_2 . Then y_2 is adjacent to 18 vertices of $X_2 \cup \{x_2, w, q\}$, hence, y_2 is adjacent to a vertex of $X_1 \cup X_3$. Since there are exactly 16 edges between X_1 and X_2 , the vertex y_2 has exactly one neighbor in X_1 . Let a be a vertex of $\Gamma(x_1, w) \cap \Gamma_2(\infty)$. By Lemma 3.4, the vertex a has a neighbor (say, b) in $X_1 \cup X_2 \cup \{u, p, x_2, q\}$. If $b \in X_1 \cup X_2$, then we have a contradiction with Lemma 3.4. If $b = u$ (or $b = x_2$), then $|\Gamma(u, w)| > 3$ (or $|\Gamma(x_1, x_2)| > 3$), a contradiction. Hence, $b \in \{p, q\}$. Since $|\Gamma(x_1, w) \cap \Gamma_2(\infty)| = 18$, we have $|\Gamma(x_1, q)| > 3$ or $|\Gamma(p, w)| > 3$, which is impossible.

At last, suppose that $X_1 \subset \Delta(u, w)$ (and $u \sim w$), $X_2 \subset \Delta(q)$ and q is adjacent to a vertex of $X_1 \cup X_3$. Let y_2 be a vertex of X_2 . Then y_2 is adjacent to 17 vertices of $X_2 \cup \{x_2, q\}$, hence, y_2 is adjacent to a couple of vertices of $X_1 \cup X_3$. Because $\Delta(y_2, x_3)$ contains q and $\Delta(y_2, u)$ contains x_2 , the vertex y_2 has exactly one neighbor in X_3 and exactly one neighbor, say y_1 , in X_1 . Now $y_1 \not\sim x_2$ and $\Delta(y_1, x_2) = \{u, w, y_2\}$, a contradiction. The lemma is proved. ■

Lemma 3.7 *Case (2) is impossible.*

Proof. We note that $|X_1| = |X_3| = 17$ and $|X_2| = 15$.

If $X_2 \subset \Delta(u, w, p, q)$, then the vertices u, w, p, q are mutually adjacent, which is impossible. If $X_1 \subset \Delta(u, w)$, then $u \sim w$ and the subgraph $\Delta(u)$ contains 17 vertices of X_1 and 3 vertices x_1, w, x_2 , hence, $|\Delta(u)| > 19$, a contradiction. So, we may assume that $X_1 \subset \Delta(u)$, $X_2 \subset \Delta(w, q)$ and, hence, $w \sim q$.

Let us consider the case $X_3 \subset \Delta(p)$. Since $\Delta(p, q) = \{x_2, x_3\}$ and the subgraph $\Delta(q)$ contains $X_2 \cup \{w, x_2, x_3\}$, the vertex q is adjacent to a vertex

of X_1 (say, q'). Symmetrically, the vertex w is adjacent to a vertex of X_3 (say, w'). A vertex $y_1 \in X_1 - \{q'\}$ is adjacent to 18 vertices of $X_1 \cup \{x_1, u\}$ and to a vertex of $X_2 \cup X_3$. Similarly, a vertex $y_3 \in X_3 - \{w'\}$ is adjacent to 18 vertices of $X_3 \cup \{x_3, p\}$ and to a vertex of $X_2 \cup X_1$. A vertex $y_2 \in X_2$ is adjacent to 17 vertices of $X_2 \cup \{x_2, w, q\}$ and to a couple of vertices of $X_1 \cup X_3$. Hence, there are the vertices, say, $a \in X_1$, $b \in X_3$ such that $a \sim b$ and each vertex $y_1 \in X_1 - \{a, q'\}$ ($y_3 \in X_3 - \{b, w'\}$, respectively) has exactly one neighbor in X_2 . Further, the vertices b, u, q induce a coclique of size 3 such that $\Delta(b, u) = \{a\}$, $\Delta(b, q) = \{x_3\}$ and $\Delta(u, q) = \{x_2, q'\}$ and this is Case (1), which is impossible.

At last, suppose that $X_2 \subset \Delta(p)$, i.e., $X_2 \subset \Delta(w, p, q)$. Then $p \sim q$ and $p \sim w$. Let y_1 be a vertex of X_1 . Then y_1 is adjacent to 18 vertices of $X_1 \cup \{x_1, u\}$, hence, y_1 is adjacent to a vertex of $X_2 \cup X_3$. Let y_3 be a vertex of X_3 . Then y_3 is adjacent to 17 vertices of $X_3 \cup \{x_3\}$, hence, y_3 is adjacent to a couple of vertices of $X_1 \cup X_2$. Let y_2 be a vertex of X_2 . Then y_2 is adjacent to 18 vertices of $X_2 \cup \{x_2, w, p, q\}$, hence, y_2 is adjacent to one vertex of $X_1 \cup X_3$. So, there are exactly 17 edges between X_1 and $X_2 \cup X_3$, exactly 15 edges between X_2 and $X_1 \cup X_3$ and exactly 34 edges between X_3 and $X_1 \cup X_2$, which is impossible. The lemma is proved. ■

Lemma 3.8 *Case (3) is impossible.*

Proof. We note that $|X_1| = |X_3| = 17$, $|X_2| = 16$ and $\Delta(u) = X_2 \cup \{x_1, x_2, x_3\}$. Hence, we have $X_1 \subset \Delta(p)$ and $X_3 \subset \Delta(q)$. Let y_1 be a vertex of X_1 . Then y_1 is adjacent to 18 vertices of $X_1 \cup \{x_1, p\}$, hence, y_1 is adjacent to a vertex of $X_3 \cup X_2$. Similarly, a vertex $y_3 \in X_3$ is adjacent to a vertex of $X_1 \cup X_2$. Let y_2 be a vertex of X_2 . Then y_2 is adjacent to 17 vertices of $X_2 \cup \{x_2, u\}$, hence, y_2 is adjacent to two vertices of $X_1 \cup X_3$. Therefore, each vertex $y_2 \in X_2$ has exactly one neighbor in X_1 and one neighbor in X_3 and there is an edge $\{a, b\}$, where $a \in X_1$, $b \in X_3$. Now the vertices b, x_1, x_2 induce a coclique of size 3 such that $\Delta(x_1, x_2) = \{p, u\}$, $\Delta(x_1, b) = \{a\}$ and $\Delta(x_2, b) = \{q\}$ and this is Case (1), a contradiction with Lemma 3.6. This contradiction completes the proof of Theorem 1.1.

References

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